28-th Bulgarian Mathematical Olympiad 1979

Fourth Round

First Day

1. Show that the equation $x^2 + 5 = y^3$ has no integer solutions.

2. Points P, Q, R, S are taken on respective edges AC, AB, BD, and CD of a tetrahedron $ABCD$ so that PR and QS intersect at point N and PS and QR intersect at point M. The line MN meets the plane ABC at point L. Prove that the lines AL, BP, and CQ are concurrent.

3. Each side of a triangle ABC has been divided into $n+1$ equal parts. Find the number of triangles with the vertices at the division points having no side parallel to or lying at a side of $\triangle ABC$.

Second Day

4. For each real number k, denote by $f(k)$ the larger of the two roots of the quadratic equation
$$(k^2+1)x^2 + 10kx - 6(9k^2+1) = 0.$$
Show that the function $f(k)$ attains its minimum and maximum and evaluate these two values.

5. A convex pentagon $ABCDE$ satisfies $AB = BC = CA$ and $CD = DE = EC$. Let S be the center of the equilateral triangle ABC and M and N be the midpoints of BD and AE, respectively. Prove that the triangles SME and SND are similar.

6. The set $M = \{1, 2, \dots, 2n\}$ $(n \geq 2)$ is partitioned into k nonintersecting subsets $M_1, M_2, \dots, M_k$, where $k^3 + 1 \leq n$. Prove that there exist $k+1$ even numbers $2j_1, 2j_2, \dots, 2j_{k+1}$ in M that are in one and the same subset M_j $(1 \leq j \leq k)$ such that the numbers $2j_1 - 1, 2j_2 - 1, \dots, 2j_{k+1} - 1$ are also in one and the same subset M_r $(1 \leq r \leq k)$.

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