Bulgarian Mathematical Olympiad 1971, IV Round

First Day

1. A natural number is called *triangled* if it may be presented in the form $\frac{n(n+1)}{2}$. Find all values of a $(1 \le a \le 9)$ for which there exist a *triangled* number all digit of which are equal to a.

2. Prove that the equation
$$\sqrt{2-x^2}+\sqrt[3]{3-x^3}=0$$
have no real solutions.

3. There are given 20 points in the plane, no three of which lies on a single line. Prove that there exist at least 969 quadrilaterals with vertices from the given points.

Second day

4. It is given a triangle ABC. Let R is the radii of the circumcircle of the triangle and O_1, O_2, O_3 are the centers of external incircles of the triangle ABC and q is the perimeter of the triangle $O_1O_2O_3$. Prove that $q \le 6\sqrt{3}R$. When does equality hold?

5. Let $A_1, A_2, \dots, A_{2n}$ are the vertices of a regular $2n$-gon and P is a point from the incircle of the polygon. If $\alpha_i = \angle A_iPA_{i+n}$, $i = 1, 2, \dots, n$. Prove the equality
$$\sum_{i=1}^{n} \tan^2 \alpha_i = 2n \frac{\cos^2 \frac{\pi}{2n}}{\sin^4 \frac{\pi}{2n}}$$

6. In a triangle pyramid $SABC$ one of the plane angles with vertex S is a right angle and orthogonal projection of S on the base plane ABC coincides with orthocentre of the triangle ABC. Let $SA = m$, $SB = n$, $SC = p$, r is the radii of incircle of ABC. H is the height of the pyramid and r_1, r_2, r_3 are radii of the incircles of the intersections of the pyramid with the plane passing through SA, SB, SC and the height of the pyramid. Prove that
   (a) $m^2+n^2+p^2 \ge 18r^2$;
   (b) $\frac{r_1}{H}, \frac{r_2}{H}, \frac{r_3}{H}$ are in the range $(0.4, 0.5)$.

Note. The last problem is proposed from Bulgaria for IMO and may be found at IMO Compendium book.

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