56-th Bulgarian Mathematical Olympiad 2007

Fourth Round

First Day – May 12

1. A circle with center I is inscribed in a quadrilateral $ABCD$ with $\angle BAD + \angle ADC > 180^\circ$. A line through I meets AB and CD at points X and Y, respectively. Prove that if $IX = IY$ then $AX \cdot DY = BX \cdot CY$.

2. Find the largest positive integer n with the following property: One can choose 2007 distinct integers from the interval $[2 \cdot 10^{n-1}, 10^n)$ such that, for any i, j with $1 \le i < j \le n$, among the selected numbers there is one, $\overline{a_1 a_2 \dots a_n}$, with $a_j \ge a_i + 2$.

3. Find the least positive integer n for which $\cos \frac{\pi}{n}$ cannot be expressed in the form $p + \sqrt{q} + \sqrt[3]{r}$ for some rational numbers p, q, r.

Second Day – May 13

4. Let $k > 1$ be an integer. A set S of natural numbers is called *good* if there is a painting of all natural numbers in k colors such that no element of S is a sum of two distinct numbers of the same color. Find the largest integer $t > 0$ for which the set $S = \{a+1, a+2, \dots, a+t\}$ is good for all $a \in \mathbb{N}$.

5. Find the least m for which any five equilateral triangles of the total area m can cover an equilateral triangle of area 1.

6. Let $f(x)$ be a monic polynomial of even degree with integer coefficients. Suppose that $f(x)$ is a perfect square for infinitely many integers x. Prove that there exists a polynomial $g(x)$ with integer coefficients such that $f(x) = g(x)^2$.

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