32-nd Bulgarian Mathematical Olympiad 1983

Third Round

First Day

1. Determine all natural nummbers m,n such that the following inequality holds:
$$\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdots\frac{m^3-1}{m^3+1}=\frac{n^3-1}{n^3+2}.$$

2. Let be given two intersecting circles $k_1(O_1,r_1)$, $k_2(O_2,r_2)$. Their common tangent meets k_1 at A and k_2 at B, and C is the common point of the circles closer to AB. The circumradius of $\triangle ABC$ is denoted as r, and $O_1O_2=d$.
   (a) Prove that if $\angle ACB=\angle O_1CO_2$, then $d^2=r_1^2+r_2^2+r^2$.
   (b) Prove that if $d^2=r_1^2+r_2^2+r^2$, then $\angle ACB=\angle O_1CO_2$.

3. A convex quadrilateral $\mathscr{K}$ is to be cut into n triangles whose vertices are at vertices or $\mathscr{K}$ or in the interior of $\mathscr{K}$, and whose every side is either a side of $\mathscr{K}$ or a side of some other triangle. Prove that:
   (a) such a cutting is possible for every even number n;
   (b) it is not possible for $n=1983$.

Second Day

4. The sequence (a_n) is defined inductively by $a_1=a_3=1$, $a_2=a_4=-1$, and $a_n=a_{n-1}a_{n-2}a_{n-4}$. Determine a_{1983}.

5. Show that the unique pair of real numbers (p,q) for which the inequality
$$\left|\sqrt{1-x^2}-px-q\right|\leq\frac{\sqrt{2}-1}{2}$$
is satisfied, is the pair $(p,q)=(-1,\frac{\sqrt{2}+1}{2})$.

6. Let be given a regular pyramid with base $ABCD$ and top V. A point M on edge BC, satisfying $BM=2MC$, is the point of BC nearest to line AV. Given that the distance from M to AV is d, find the volume of the pyramid.