Bulgarian Mathematical Olympiad 1976, III Round

First Day

1. Let α is a positive number. Prove that
$$\sqrt[3]{27+8\alpha} < \sqrt[3]{1+\alpha} + \sqrt[3]{8+\alpha}$$
(8 points, J. Tabov)

2. In a triangle pyramid $MABC$: $\angle AMB = \angle BMC = \angle CMA = 90^\circ$. If h is the height of the pyramid from M to ABC, r is the radii of inscribed in the pyramid sphere and V is its volume. Prove that
$$V \geq \frac{9h^3r^3}{2(h-r)^3}$$
where eqyuality occurs only if the three edges passing through M are all equal to each other. (7 points)

3. Let natural numbers m and n are sidelengths of the rectangle P divided to mn squares with side 1 with lines parallel to its sides. Prove that the diagonal of the rectangle passes through internal points of $m+n-d$ squares, when d is the biggest common divisor of m and n.
(6 points, I. Tonov)

Second day

4. It is given the system
$$\begin{cases} x^2 - |x|\sqrt{y} - y\sqrt{y} = 0 \\ x^2 - x\left(y + 2y\sqrt{y}\right) + y^3 = 0 \end{cases}$$
Find all real solutions (x,y) of the system for which $x \neq 1$. (8 points)

5. Let the point P is internal for the circle k with center O. The cord AB is passing through P. Prove that $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$, where $\alpha = \angle AOP$, $\beta = \angle BOP$ is a costant for all chords passing through P. (4 points)

6. In an acute angled triangle through its orthocentre is drawn a line ℓ which intersects two sides of the triangle and the continuation of third side. There are drawn the lines symmetric of ℓ about the sides of the triangle. Prove that these lines intersects at a common point from circumscribed circle of the triangle ABC. (7 points, V. Petkov)

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
LaTeX by Borislav Mirchev
www.imomath.com