Bulgarian Mathematical Olympiad 1971, III Round

First Day

1. Prove that the equation
$$x^{12} - 11y^{12} + 3z^{12} - 8t^{12} = 1971^{1970}$$
don't have solutions in integer numbers. (5 Points)

2. Solve the system:
$$\text{(a)} \begin{cases} x = \dfrac{2y}{1+y^2} \\ y = \dfrac{2z}{1+z^2} \\ z = \dfrac{2x}{1+x^2} \end{cases} \qquad \text{(b)} \begin{cases} x = \dfrac{2y}{1-y^2} \\ y = \dfrac{2z}{1-z^2} \\ z = \dfrac{2x}{1-x^2} \end{cases}$$
(x, y, z are real numbers). (7 Points)

3. Let E is a system of 17 segments over a straight line. Prove:
   (a) or there exist a subsystem of E that consist from 5 segments which on good satisfying ardering includes monotonically in each one (the first on the second, the second on the next and ect.)
   (b) or can be found 5 segments from ε, no one of them is contained in some of the other 4.

   (8 Points)

Second day

4. Find all possible conditions for the real numbers a, b, c for which the equation $a\cos x + b\sin x = c$ have two solutions, x' and x'', for which the difference $x' - x''$ is not divisible by π and $x' + x'' = 2k\pi + \alpha$ where α is a given number and k is an integer number. (6 Points)

5. Prove that if in a triangle two of three angle bisectors are equal the triangle is isosceles. (6 Points)

6. It is given a cube with edge a. On distance $\frac{a\sqrt{3}}{8}$ from the center of the cube is drawn a plane perpendicular to some of diagonals of the cube:
   (a) find the shape/kind of the intersection of the plane with the cube;
   (b) calculate the area of this intersection.

   (8 Points)

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