Bulgarian Mathematical Olympiad 1962, III Round

First Day

1. It is given the sequence: $1, 1, 2, 3, 5, 8, 13, \ldots$, each therm of which after the second is equal to the sum of two terms before it. Prove that the absolute value of the difference between the square of each term from the sequence and multiple of the term before it and the term after it is equal to 1. (7 points)

2. Find the solutions of the inequality:

$$\sqrt{x^2 - 3x + 2} > x - 4$$

(7 points)

3. For which triangles the following equality is true:

$$\cos^2 \alpha \cot \beta = \cot \alpha \cos^2 \beta$$

(6 points)

Second day

4. It is given the angle $\angle XOY = 120^\circ$ with angle bisector OT. From the random point M chosen in the angle $\angle TOY$ are drawn perpendiculars MC, MA and MB respectively to OX, OY and OT. Prove that:

   (a) triangle ABC is equilateral;

   (b) the following relation is true: $MC = MA + MB$;

   (c) the surface of the triangle ABC is $S = \frac{\sqrt{3}}{4}\left(a^2 + ab + b^2\right)$, where $MA = a$, $MB = b$.

   (7 points)

5. On the base of isosceles triangle ABC is chose a random point M. Through M are drawn lines parallel to the non-base sides, intersecting AC and BC respectively at the points D and E:

   (a) prove that: $CM^2 = AC^2 - AM \cdot BM$;

   (b) find the locus of the feets to perpendiculars drawn from the centre of the circumcircle over the triangle ABC to diagonals MC and ED of the parallelogram $MECD$ when M is moving over the base AB;

   (c) prove that : $CM^2 = AC^2 - AM \cdot BM$ if M is over the extension of the base AB of the triangle ABC.

   (7 points)

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6. What is the distance from the centre of a sphere with radii R for which a plane must be drawn in such a way that the full surface of the pyramid with vertex same as the centre of the sphere and base square which is inscribed in the circle formed from intersection of the sphere and the plane is 4 m^2.

(6 points)

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