Brazilian IMO & IbMO Team Selection Tests 1998

First Test – March 8, 1998

Time: 4.5 hours

1. Let N be a positive integer greater than 2. We number the vertices of a regular $2n$-gon clockwise with the numbers $1,2,\dots,N,-N,-N+1,\dots,-2,-1$. Then we proceed to mark the vertices in the following way.

   In the first step we mark the vertex 1. If n_i is the vertex marked in the i-th step, in the $i+1$-th step we mark the vertex that is $|n_i|$ vertices away from vertex n_i, counting clockwise if n_i is positive and counter-clockwise if n_i is negative. This procedure is repeated till we reach a vertex that has already been marked. Let $f(N)$ be the number of non-marked vertices.

   (a) If $f(N)=0$, prove that $2N+1$ is a prime number.

   (b) Compute $f(1997)$.

2. Let S be a finite set of real numbers with the property that any two distinct elements of S form an arithmetic progression with another element from S. Find such a set S with 5 elements and prove that S cannot have more than five elements.

3. Let $\mathbb{N}$ be the set of positive integers. Find all functions defined on $\mathbb{N}$ and taking values on $\mathbb{N}$ satisfying, for all $n\in\mathbb{N}$,

$$f(n)+f(n+1)=f(n+2)f(n+3)-1998.$$

4. Let L be a circle with center O and tangent to sides AB and AC of a triangle ABC in points E and F, respectively. Let the perpendicular from O to BC meet EF at D. Prove that A,D and M are collinear, where M is the midpoint of BC.

5. Consider k positive integers $a_1,a_2,\dots,a_k$ satisfying $1\le a_1<a_2<\dots<a_k\le n$ and $\operatorname{lcm}(a_i,a_j)\le n$ for any i,j. Prove that

$$k\le 2\left[\sqrt{n}\right].$$

Second Test – May 16, 1998

1. Let ABC be an acute-angled triangle. Construct three semi-circles, each having a different side of ABC as diameter, and outside ABC. The perpendiculars dropped from A,B,C to the opposite sides intersect these semi-circles in points E,F,G, respectively. Prove that the hexagon $AGBECF$ can be folded so as to form a pyramid having ABC as base.

www.imomath.com

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

2. There are $n \geq 3$ integers around a circle. We know that for each of these numbers the ratio between the sum of its two neighbors and the number is a positive integer. Prove that the sum of the n ratios is not greater than $3n$.

3. Show that it is possible to color the points of $\mathbb{Q} \times \mathbb{Q}$ in two colors in such a way that any two points having distance 1 have distinct colors.

4. (a) Show that, for each positive integer n, the number of monic polynomials of degree n with integer coefficients having all its roots on the unit circle is finite.

   (b) Let $P(x)$ be a monic polynomial with integer coefficients having all its roots on the unit circle. Show that there exists a positive integer m such that $y^m = 1$ for each root y of $P(x)$.

5. Let p be an odd prime integer and k a positive integer not divisible by p, $1 \leq k < 2(p+1)$, and let $N = 2kp+1$. Prove that the following statements are equivalent:

   (i) N is a prime number;

   (ii) there exists a positive integer a, $2 \leq a < n$, such that $a^{kp}+1$ is divisible by N and $(a^k+1, N) = 1$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com