Brazilian IMO & IbMO Team Selection Tests 1997

First Test

Time: 4.5 hours

1. Let ABC be a triangle and L its circumscribed circle. The internal bisector of angle A meets BC at point P. Let L_1 be the circle tangent to AP, BP and L. Similarly, let L_2 be the circle tangent to AP, CP and L. Prove that the tangency points of L_1 and L_2 with AP coincide.

2. We say that a subset A of $\mathbb{N}$ is *good* if for some positive integer n the equation $x-y=n$ admits infinitely many solutions with $x,y \in A$. If $A_1, A_2, \dots, A_{100}$ are sets whose union is $\mathbb{N}$, prove that at least one of the A_is is good.

3. Determine all positive integers x, y $(x > 1)$ and all primes p and q satisfying $p^x = 2^y + q^x$.

4. Prove that it is impossible to arrange the numbers $1, 2, \dots, 1997$ around a circle in such a way that, if x and y are any two neighboring numbers, then $499 \le |x-y| \le 997$.

5. Let ABC be an acute-angled triangle with incenter I. Consider the point A_1 on AI different from A, such that the midpoint of AA_1 lies on the circumscribed circle of ABC. Points B_1 and C_1 are defined similarly.

   (a) Prove that $S_{A_1B_1C_1} = (4R+r)p$, where p is the semi-perimeter, R is the circumradius and r is the inradius of ABC.

   (b) Prove that $S_{A_1B_1C_1} \ge 9S_{ABC}$.

Second Test

1. In an isosceles triangle ABC $(AC = BC)$, let O be its circumcenter, D the midpoint of AC and E the centroid of DBC. Show that OE is perpendicular to BD.

2. Prove that any group of people can be divided into two disjoint groups A and B such that any member from A has at least half of his acquaintances in B and any member from B has at least half of his acquaintances in A (acquaintance is reciprocal).

3. Let b be a positive integer such that $(b,6) = 1$. Show that there are positive integers x and y such that $\frac{1}{x} + \frac{1}{y} = \frac{3}{b}$ if and only if b is divisible by some prime number of form $6k-1$.

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4. Consider an $N \times N$ matrix, where N is an odd positive integer, such that all its entries are $-1, 0$ or 1. Consider the sum of the numbers in every line and every column. Prove that at least two of the $2N$ sums are equal.

5. Consider an infinite strip, divided into unit squares. A finite number of nuts is placed in some of these squares. In a step, we choose a square A which has more than one nut and take one of them and put it on the square on the right, take another nut (from A) and put it on the square on the left. The procedure ends when all squares has at most one nut. Prove that, given the initial configuration, any procedure one takes will end after the same number of steps and with the same final configuration.

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