Brazilian IMO & IbMO Team Selection Tests 2001

First Test – March 24, 2001

Time: 4.5 hours

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying
$$f(x+y)+f(y+z)+f(z+x) \geq f(x+2y+3z)$$
for all real x,y,z.

2. Let $f(n)$ be the least positive integer k such that n divides $1+2+\cdots+k$. Prove that $f(n) = 2n-1$ if and only if n is a power of 2.

3. For which positive integers n is there a permutation $(x_1,x_2,\dots,x_n)$ of $1,2,\dots,n$ such that all the differences $|x_k - k|$, $k = 1,2,\dots,n$, are distinct?

4. Let ABC be a triangle with the circumcenter at O. Let P,Q be points on the segments AB and AC respectively so that
$$BP : PQ : QC = AC : CB : BA.$$
Prove that the points A,P,Q and O are concyclic.

Second Test – May 19, 2001

1. Polynomials $P(x)$ and $Q(x)$ with real coefficients, both of which having at least one real root, satisfy the equality
$$P(1+x+Q(x)^2) = Q(1+x+P(x)^2)$$
for all real x. Prove that the polynomials P and Q are equal.

2. A set S consists of k sequences of $0,1,2$ of length n. For any two sequences $(a_i),(b_i) \in S$ we can construct a new sequence (c_i) such that $c_i = \left[\dfrac{a_i+b_i+1}{2}\right]$ and include it in S. Assume that after performing finitely many such operations we obtain all the 3^n sequences of $0,1,2$ of length n. Find the least possible value of k.

3. Let ABC be a triangle and D,E be the points of intersection of the internal and external bisectors of the angle at A with BC. Let $F \neq A$ be the intersection point of line AC with the circle with diameter DE. Let $G \neq A$ be the point at which the tangent at A on the circumcircle of ABF meets the circle with diameter DE. Prove that $AF = AG$.

4. Prove that for all integers $n \geq 3$ there exists a set $A_n = \{a_1,a_2,\dots,a_n\}$ of n distinct natural numbers such that, for each $i = 1,2,\dots,n$,
$$\prod_{\substack{1\le k\le n\\ k\neq i}} a_k \equiv 1 \mod a_i.$$

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com