19-th Brazilian Mathematical Olympiad 1997

Final Round

First Day

1. Two circles with centers at O and O' and radii R and r, respectively, intersect at points P and P'. Let l be the line through P and P'. Determine the smallest possible sum of the distances from O and O' to l in terms of R and r.

2. We say that a set $A \subset \mathbb{N}$ has the property $P(n)$ if A has n elements and $A+A := \{x+y \mid x,y \in A\}$ has exactly $\frac{n(n+1)}{2}$ elements. For each finite subset A of $\mathbb{N}$ we define its diameter as the difference between the largest and the smallest element of A. Let $f(n)$ be the smallest possible diameter of a set A satisfying $P(n)$. Prove that
$$\frac{n^2}{4} \leq f(n) < n^3 \qquad \text{for all } n \geq 2.$$
(If you still have time, try to improve this estimate. For example, try to show that $f(p) < 2p^2$ for every prime p.)

3. (a) Prove that there are no functions $f,g : \mathbb{R} \to \mathbb{R}$ satisfying
$$g(f(x)) = x^3 \text{ and } f(g(x)) = x^2 \qquad \text{for all } x \in \mathbb{R}.$$
(b) Give an example of functions $f,g : (1,\infty) \to (1,\infty)$ satisfying
$$g(f(x)) = x^3 \text{ and } f(g(x)) = x^2 \qquad \text{for all } x > 1.$$

Second Day

4. Let F_n be defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$ for all $n \geq 1$. Let
$$V_n = \sqrt{F_n^2 + F_{n+2}^2} \qquad \text{for } n \geq 1.$$
Prove that for each n, V_n, V_{n+1} and V_{n+2} are sides of a triangle of area $1/2$.

5. Set $f(x) = x^2 + c$, where $c \in \mathbb{Q}$. Define $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for each $n \in \mathbb{N}$. We say that $x \in \mathbb{R}$ is *pre-periodic* if the set $\{f^n(x) \mid n \in \mathbb{N}\}$ is finite. Prove that the set $\{x \in \mathbb{Q} \mid x \text{ is pre-periodic}\}$ is finite.

6. Suppose that a mapping f from a plane to itself satisfies
$$d(f(P), f(Q)) = 1 \qquad \text{whenever} \qquad d(P,Q) = 1.$$
Prove that $d(f(P), f(Q)) = d(P,Q)$ for all points P,Q in the plane. ($d(X,Y)$ denotes the distance between X and Y.)

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