17-th Brazilian Mathematical Olympiad 1995

Final Round

First Day

1. Let $ABCD$ be an inscribable and circumscribable quadrilateral, and let I, O and S be respectively the incenter, circumcenter and the intersection point of the diagonals. Prove that if two of the three points I, O, S coincide, then $ABCD$ is a square.

2. Find the number of functions $f : \mathbb{N}_0 \to \mathbb{R}$ satisfying:

   (i) $f(x+1019) = f(x)$ for all x;

   (ii) $f(xy) = f(x)f(y)$ for all $x, y \in \mathbb{N}_0$.

3. Let $P(n)$ denote the greatest prime divisor of a natural number $n > 1$. Prove that there are infinitely many n for which
$$P(n) < P(n+1) < P(n+2).$$

Second Day

4. A regular tetrahedron of side length x is given. A student wants to make a closed curve out of a string (which may change its shape, but maintaining the closedness and the length), such that the tetrahedron may pass through the string. Find the minimum possible length of the string.

5. Prove that no zero of the polynomial
$$x^5 - x^4 - 4x^3 + 4x^2 + 2$$
is an n-th root of a rational number.

6. Let X be a set of n elements, and let $\mathscr{F}$ be a set of three-element subsets of X such that any two sets in $\mathscr{F}$ have at most one element in common. Show that there is a subset of X containing at least $[\sqrt{2n}]$ elements and not containing any set from $\mathscr{F}$.

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