

30-th Brazilian Mathematical Olympiad 2008

Third Round

First Day

1. A positive integer is called *nice* if at least one of its multiples starts with 2008 in decimal representation. For example, 7 is nice because $200858 = 28694 \cdot 7$. Prove that every positive integer is *nice*.
2. Let S be a subset of a line consisting of $6n$ points. $4n$ points are chosen at random from S and painted in blue. The remaining $2n$ points are painted in red. Prove that there exists a line segment that contains exactly $3n$ points from S , $2n$ of which are blue and n of which are red.
3. Let x, y, z be real numbers such that $x + y + z = xy + yz + zx$. Find the minimal value of

$$\frac{x}{x^2 + 1} + \frac{y}{y^2 + 1} + \frac{z}{z^2 + 1}.$$

Second Day

4. Given a cyclic quadrilateral $ABCD$, let r and s be the lines obtained by reflecting AB with respect to the internal bisectors of $\angle CAD$ and $\angle CBD$. If P is the intersection point of the lines r and s and O is the circumcenter of $ABCD$ prove that $OP \perp CD$.
5. Prove that for all integers $a > 1$ and $b > 1$ there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(a \cdot f(n)) = b \cdot n$ for all $n \in \mathbb{N}$.
6. A Venusian Prophet sent to his people a 10000-letter word consisting of letters A and E only. For each $k \in \{1, 2, \dots, 10000\}$ the people made all possible words formed by taking k consecutive letters from the word obtained by the Prophet. It is known that people obtained at most seven words of length 3. Find the maximal possible number of words of length 10 that the people could obtain.