Belarusian Team Selection Test 1995

First Day

1. There is a 100×100 square table, a real number being written in each cell. A and B play the following game. They choose, turn by turn, some row of the table (if it has not been chosen before). When A and B have 50 rows chosen each, they sum the numbers in the corresponding cells of the chosen rows, and then sum the squares of all 100 obtained numbers and compare the results. A player who has the greater result wins. Player A begins. Show that A can avoid a defeat.

2. Circles S, S_1, S_2 are given in a plane. S_1 and S_2 touch each other externally, and both touch S internally at A_1 and A_2 respectively. The common internal tangent to S_1 and S_2 meets S at P and Q. Let B_1 and B_2 be the intersections of PA_1 and PA_2 with S_1 and S_2, respectively. Prove that B_1B_2 is a common tangent to S_1, S_2.

3. Show that there is no infinite sequence a_n of natural numbers such that
$$a_{a_n} = a_{n+1}a_{n-1} - a_n^2 \quad \text{for all } n \geq 2.$$

Second Day

4. Prove that the number of odd coefficients in the polynomial $(1+x)^n$ is a power of 2 for every positive integer n.

5. There is a room having a form of right-angled parallelepiped. Four maps of the same scale are hung (generally, on different levels over the floor) on four walls of the room, so that sides of the maps are parallel to sides of the wall. It is known that the four points corresponding to each of Stockholm, Moscow, and Istanbul are coplanar. Prove that the four points coresponding to Hong Kong are coplanar as well.

6. If $0 < a, b < 1$ and $p, q \geq 0$, $p+q = 1$ are real numbers, prove that
$$a^p b^q + (1-a)^p (1-b)^q \leq 1.$$

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com