49-th Belarusian Mathematical Olympiad 1999

Final Round

Category D

First Day

1. Twenty counters are arranged in a circle. Two players alternately remove counters, arbitrary three in a move, until only two remain. If the two remaining counters were not adjacent in the initial arangement, then the first player (who played first) wins; otherwise, the second player wins. Determine who of the players has a winning strategy.

2. Let k be the ratio of the roots of the equation $px^2 - qx + q = 0$, where $p, q > 0$. Find the roots of the equation $\sqrt{p}x^2 - \sqrt{q}x + \sqrt{p} = 0$ in terms of k only.

3. Find the least integer $n \geq 9$ with the following property: It is possible to choose nine of the numbers $1, 2, \ldots, n$ and arrange them in the cells of a 3×3 board, so that in any two neighboring cells one of the numbers is divisible by the other one.

4. Let $ABCDE$ be a pentagon with $AE = ED$, $AB + CD = BC$, and $\angle BAE + \angle CDE = 180^\circ$. Prove that $\angle AED = 2\angle BEC$.

Second Day

5. Find the number of positive divisors of 10^{999} that are not divisors of 10^{998}.

6. Let ABC be a right isosceles triangle $AC = BC$ and M be the midpoint of AB. Points K and N are taken on the sides BC and AC respectively so that $BK : KC = AN : NC = 2 : 1$. Lines AK and MN meet at L. Prove that LC bisects the angle KLN.

7. During a conference, participants gave each other souvenirs. It turned out that everyone got exactly one souvenir from every other participant. Moreover, at some moment during the conference at least half of the participants got exactly half of the souvenirs, at least one third of the participants got exactly one third of the souvenirs, and at least one seventh of the participants got exactly one seventh of the souvenirs. What is the least possible number of participants at the conference?

8. Let n and m be positive integers. A positive integer is said to be *attainable* if it is 1 or can be obtained from 1 by a sequence of operations with the following properties:

   (i) The first operation is either addition or multiplication.

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(ii) Thereafter, additions and multiplications are used alternately.

(iii) In each addition one can choose independently whether to add n or m.

(iv) In each multiplication, one can choose independently whether to multiply by n or by m.

A positive integer that cannot be so obtained is said to be *unattainable*. Prove that if $n = 3$ and $m \geq n$, then there are infinitely many unattainable positive integers.

Category C

First Day

1. On an island there is a one-way connection between any two islanders. Every day, each islander contrives a new gossip and tells it to whoever on the island he can, along with all the other gossips that he received in the previous day. Prove that if there is an islander who didn't receive on Wednesday a gossip transmitted by him on Monday, then there is a gossip that will never be transmitted to at least one of the islanders.

2. A finite set M consisting of at least two positive real numbers has the property that for any $a \in M$ there exist numbers $b, c \in M$ (a, b, c are not necessarily distinct) such that $a = 1 + b/c$. Prove that there are two different numbers $x, y \in M$ such that $x + y > 4$.

3. The rows and columns of an $n \times n$ board are enumerated by 1 to n. In each cell of the board is written either -1 or 1.

   (a) Find all n for which it is possible to fill the board so that in any row and column that are enumerated by the same number the products of entries are different.

   (b) For all such n, find the smallest number of -1-s for which this is possible.

4. In an acute-angled triangle ABC, the circle with diameter AB intersects CA at L and CB at N. The segment LN intersects the median CM at K. Compute CM, given that $AB = 9$ and $CK = \frac{3}{5}CM$.

Second Day

5. Suppose that nonzero numbers a, b, c satisfy the equalities $a^2 - b^2 = bc$ and $b^2 - c^2 = ca$. Prove that $a^2 - c^2 = ab$.

6. At least 32 students took part at a mathematical olympiad. It is known that any number of students exceeding half of the participants have at least 50 points in total. All participants have 150 points in total, and there are no negative or non-integer numbers of points. What is the largest possible number of points the winner can have?

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7. Let AB and CD be perpendicular bisectors of a circle, and K be a point on the circle other than A,B,C,D. Let the lines AK and CD meet at M, and the lines DK and BC meet at N. Prove that MN is parallel to AB.

8. Let n and m be positive integers. A positive integer is said to be *attainable* if it is 1 or can be obtained from 1 by a sequence of operations with the following properties:

   (i) The first operation is either addition or multiplication.

   (ii) Thereafter, additions and multiplications are used alternately.

   (iii) In each addition one can choose independently whether to add n or m.

   (iv) In each multiplication, one can choose independently whether to multiply by n or by m.

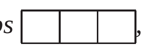
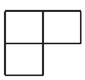
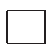
   A positive integer that cannot be so obtained is said to be *unattainable*. Prove that if $m \geq n \geq 4$, then there are infinitely many unattainable positive integers.

Category B

First Day

1. Find all real numbers a for which the function $f(x) = \{ax + \sin x\}$ is periodic.

2. Prove that is k is the number of divisors of an integer $N > 1$ (including 1 and N) and S their sum, then $k\sqrt{N} < S < \sqrt{2kN}$.

3. A 7×7 board and tiles of the following three types: *strips* , *corners*  and *squares*  are given. Jerry has infinitely many strips and only one corner, while Tom has only one square.

   (a) Prove that Tom can put his tile on the board so that Jerry cannot tile the rest of the board with his tiles.

   (b) Suppose that Jerry acquired another corner, so that he has many strips and two corners. Prove that no matter on which cell Tom puts his tile, Jerry can tile the rest of the board.

4. A circle is inscribed in an isosceles trapezoid $ABCD$. The diagonal AC intersects the circle at K and L, in the order A,K,L,C. Find the value of $\sqrt[4]{\dfrac{AL \cdot KC}{AK \cdot LC}}$.

Second Day

5. Let P and Q be points on the side AB of a triangle ABC such that $\angle ACP = \angle PCQ = \angle QCB$. The bisector AD of $\angle BAC$ meets CP and CQ at M and N respectively. Prove that if $PN = CD$ and $3\angle A = 2\angle C$, then $S_{CQD} = S_{QNB}$.

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6. Prove that the equation $\{x^3\}+\{y^3\}=\{z^3\}$ has infinitely many rational non-integer roots.

7. Find all $n\in\mathbb{N}$ and $m\in\mathbb{R}$ with the following property: One can write the numbers $1,2,\dots,n^2$ in the cells of an $n\times n$ board and enumerate its rows and columns by $1,\dots,n$ in such a way that

$$(m-1)a\le(i+j)^2-(i+j)\le ma$$

holds for any $i,j\in\{1,\dots,n\}$, where a is the number in the intersection of the i-th row and the j-th column.

8. Let n be a positive integer. A positive integer is said to be *attainable* if it is 1 or can be obtained from 1 by a sequence of operations with the following properties:

   (i) The first operation is either addition or multiplication.

   (ii) Thereafter, additions and multiplications are used alternately.

   (iii) In each addition one can choose independently whether to add 2 or n.

   (iv) In each multiplication, one can choose independently whether to multiply by 2 or by n.

   A positive integer that cannot be so obtained is said to be *unattainable*. Prove that if $n\ge 9$, then there are infinitely many unattainable positive integers.

Category A

First Day

1. Evaluate the product $\displaystyle\prod_{k=0}^{2^{1999}}\left(4\sin^2\frac{k\pi}{2^{2000}}-3\right)$.

2. Let m,n be positive integers. Starting with all positive integers written in a line, we can form a list of numbers in two ways:

   (1) Erasing every m-th number and then, in the obtained list, erasing every n-th number;

   (2) Erasing every n-th number and then, in the obtained list, erasing every m-th number.

   A pair (m,n) is called *good* if, whenever some positive integer k occurs in both these lists, then it occurs in both lists on the same position.

   (a) Show that the pair $(2,n)$ is good for any $n\in\mathbb{N}$.

   (b) Is there a good pair (m,n) with $2<m<n$?

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3. A sequence of numbers $a_1, a_2, \dots, a_{1999}$ is given. In each move it is allowed to choose two of the numbers, say a_m, a_n, and replace them by the numbers

$$\frac{a_n^2}{a_m^2} - \frac{n}{m}\left(\frac{a_m^2}{a_n} - a_m\right), \quad \frac{a_m^2}{a_n^2} - \frac{m}{n}\left(\frac{a_n^2}{a_m} - a_n\right),$$

respectively. Starting with the sequence $a_i = 1$ for $20 \nmid i$ and $a_i = \frac{1}{5}$ for $20 \mid i$, is it possible to obtain a sequence whose all terms are integers?

4. A circle is inscribed in a trapezoid $ABCD$ and intersects the diagonal AC at K, L and the diagonal BD at M, N, in the orders A, K, L, C and B, M, N, D. Given that $AK \cdot LC = 16$ and $BM \cdot ND = 2.25$, compute the radius of the circle.

Second Day

5. Find the greatest real number k such that, whenever positive real numbers a, b, c satisfy $kabc > a^3 + b^3 + c^3$, there exists a triangle with sides a, b, c.

6. Find all integers x and y such that $x^6 + x^3y = y^3 + 2y^2$.

7. Let O be the center of a circle S. Two equal chords AB and CD of S intersect at L, where $AL > LB$ and $DL > LC$. Let M and N be points on the segments AL and DL such that $\angle ALC = 2\angle MON$. Prove that the chord of S passing through M and N is equal to AB and CD.

8. A positive integer is said to be *attainable* if it is 1 or can be obtained from 1 by a sequence of operations with the following properties:

   (i) The first operation is either addition or multiplication.

   (ii) Thereafter, additions and multiplications are used alternately.

   (iii) In each addition one can choose independently whether to add 2 or 3.

   (iv) In each multiplication, one can choose independently whether to multiply by 2 or by 3.

   Prove that all positive integers except 7 are attainable.