47-th Belarusian Mathematical Olympiad 1997

Final Round

Category D

First Day

1. A two-digit number which is not a multiple of 10 is given. Assuming it is divisible by the sum of its digits, prove that it is also divisible by 3. Does the statement hold for three-digit numbers as well?

2. Points D and E are taken on side CB of triangle ABC, with D between C and E, such that $\angle BAE = \angle CAD$. If $AC < AB$, prove that $AC \cdot AE < AB \cdot AD$.

3. Is it possible to mark 10 red, 10 blue and 10 green points on a plane such that: For each red point A, the point (among the marked ones) closest to A is blue; for each blue point B, the point closest to B is green; and for each green point C, the point closest to C is red?

4. The sum of 5 positive numbers equals 2. Let S_k be the sum of the k-th powers of these numbers. Determine which of the numbers $2, S_2, S_3, S_4$ can be the greatest among them.

Second Day

5. Find all composite numbers n with the following property: For every proper divisor d of n (i.e. $1 < d < n$), it holds that $n - 20 \leq d \leq n - 12$.

6. If distinct real numbers x, y satisfy $\{x\} = \{y\}$ and $\{x^3\} = \{y^3\}$, prove that x is a root of a quadratic equation with integer coefficients.

7. If $ABCD$ is as convex quadrilateral with $\angle ADC = 30^\circ$ and $BD = AB + BC + CA$, prove that BD bisects $\angle ABC$.

8. Straight lines k, l, m intersecting each other in three different points are drawn on a classboard. Bob remembers that in some coordinate system the lines k, l, m have the equations $y = ax$, $y = bx$ and $y = c + \frac{2ab}{a+b}x$ (where $ab(a+b) \neq 0$). Misfortunately, both axes are erased. Also, Bob remembers that there is missing a line n ($y = -ax + c$), but he has forgotten a, b, c. How can he reconstruct the line n?

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Category C

First Day

1. A positive integer N is a sum of two nonzero squares m^2 and n^2. Prove that none of divisors of N can be written in the form $m^k - n^k$ for $k \geq 2$.

2. If two-digit numbers a and b have the same digits in reverse order, find the least possible value of $|\frac{a}{b} - 2|$.

3. Points D, M, N are chosen on the sides AC, AB, BC of a triangle ABC respectively, so that the intersection point P of AN and CM lies on BD. Prove that BD is a median of the triangle if and only if $AP : PN = CP : PM$.

4. An equilateral triangle ABC of side n is divided into n^2 equilateral triangles of side 1. All vertices of small triangles is labeled with numbers - A, B, C, and a point D on the distance $\sqrt{3}$ from C are labelled by 1, and all other points by 0. In each move it is allowed to increase or decrease by 1 all the four vertices of a rhombus of side 1. Find all numbers $n \geq 3$ for which we can make all the numbers vanish.

Second Day

5. Prove that there exist infinitely many triples (a, b, c) of integers satisfying
$$a^3 - 3ab - b^3 = b^3 + 3bc + c^3 = 1.$$

6. A finite number of lines are drawn in a plane. Determine whether the following statement is true: There always exists a line a among the drawn lines such that all intersection points of the lines are in the same halfplane determined by a, or on a.

7. We are given a mechanism that can perform the following operations:
   - Joining any two points of a plane by a straight line;
   - Constructing the reflection X of a given point P in a given line l.

   Given a triangle ABC, using the given mechanism, construct (a) its centroid; (b) its circumcenter.

8. An archipelago consists of n islands. Some of the islands are pairwise connected by bridges (maybe zero or more than one). It is known that from any island one can arrive to any other island, and that the first island has 1 bridge, the second 4, $\dots$, the n-th n^2 bridges. Find all n for which such an archipelago can exist.

Category B

First Day

1. A pentagon $A_1A_2A_3A_4A_5$ is inscribed in a circle, B being the intrersection point of A_1A_4 and A_2A_5. Given that $\angle A_4A_1A_3 = \angle A_5A_2A_4$ and $\angle A_2A_4A_1 = \angle A_3A_5A_2$, prove that $\angle A_1A_3B = \angle BA_3A_5$.

2. Let M be the set $\{0,1,2,4,5,6,...,n\}$ and let $s : M \to M$ be a bijection. Find, for various n, all possible values of the greatest common divisor of the numbers $i+s(i)$, $i \in M$.

3. In a solving riddles contest, 5 riddles were proposed. After the contest, every jury member assigned to each riddle some positive integer number of points, and determined a winner. Curiously, every contestant was determined as a winner by at least one jury member. Find the greatest possible number of participants (a riddle is either solved (full points) or unsolved (0)).

4. On each cell of a 8×8 chessboard bthere is a white or black pawn. It is known that any black pawn has an odd number of white neighbours, while any white pawn has an even number of black neighbours (two pawns are neighboring whenever their cells share a vertex).

   (a) Find the greatest possible number of black pawns.

   (b) Prove that the central 4×4 square contains an even number of black pawns.

Second Day

5. In a trapezoid $ABCD$ with $AB \parallel CD$ it holds that $\angle ADB + \angle DBC = 180^\circ$. Prove that $AB \cdot BC = AD \cdot DC$.

6. There are n red and n blue points on a straight line l. Prove that there is a point M on l such that the sum of distances from M to the blue points is equal to the sum of distances from M to the red points. Is this statement true if the n blue and n red points are arbitrarily given in a plane?

7. The polynomials $x, x^3, x^5, \ldots$ are written on a classboard. New polynomials are written according to the following rule: if $f(x)$ and $g(x)$ have been already written, one can write any of the polynomials $f(x)+g(x)$, $f(g(x))$ and $af(x)+b$ for any $a,b \in \mathbb{R}$. Is it possible to obtain a polynomial of the form $x^{2k+1} - 19x + 97$, where k is a positive integer?

8. The sum of n real numbers is positive. Find the smallest possible number of their pairwise sums that must be positive.

Category A

First Day

1. Different points A_1, A_2, A_3, A_4, A_5 lie on a circle so that $A_1A_2 = A_2A_3 = A_3A_4 = A_4A_5$. Let A_6 be the diametrically opposite point to A_2, and A_7 be the intersection of A_1A_5 and A_3A_6. Prove that the lines A_1A_6 and A_4A_7 are perpendicular.

2. A sequence $(a_n)_{n=-\infty}^{\infty}$ of zeros and ones is given. It is known that $a_n = 0$ if and only if $a_{n-6} + a_{n-5} + \cdots + a_{n-1}$ is a multiple of 3, and not all terms of the sequence are zero. Determine the maximum possible number of zeros among $a_0, a_1, \dots, a_{97}$.

3. If a, x, y, z are positive numbers, prove the inequality
$$x\frac{a+z}{a+x} + y\frac{a+x}{a+y} + z\frac{a+y}{a+z} \le x+y+z \le x\frac{a+y}{a+z} + y\frac{a+z}{a+x} + z\frac{a+x}{a+y}.$$

4. A set M consists of n elements. Find the greatest k for which there is a collection of k subsets of M with the following property: For any subsets $A_1, \dots, A_j$ from the collection there is an element belonging to an odd number of them.

Second Day

5. We call the sum of any k of n given numbers (with distinct indices) a k-sum. Given n, find all k such that, whenever more than half of k-sums of numbers $a_1, \dots, a_n$ are positive, the sum $a_1 + \cdots + a_n$ is also positive.

6. Suppose that a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies
$$f(f(x)) + x = f(2x).$$
Prove that $f(x) \ge x$ for all x.

7. Does there exist an infinite set M of straight lines on the coordinate plane such that

   (i) no two lines are parallel, and

   (ii) for any integer point there is a line from M containing it?

   Does the answer change if we add one more condition:

   (iii) any line from M passes through at least 2 integer points?

8. A triangle $A_1B_1C_1$ is a parallel projection of a triangle ABC in space. The parallel projections A_1H_1 and C_1L_1 of the altitude AH and the bisector CL of $\triangle ABC$ respectively are drawn. Using a ruler and compass, construct a parallel projection of (a) the orthocenter; (b) the incenter of $\triangle ABC$.

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