50-th Belarusian Mathematical Olympiad 2000

Final Round

Category D

First Day

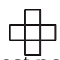
1. Find all pairs of integers (x,y) satisfying $3xy - x - 2y = 8$.

2. Points M and K are marked on the sides BC and CD of a square $ABCD$, respectively. The segments MD and BK intersect at P. Prove that $AP \perp MK$ if and only if $MC = KD$.

3. The roots of the quadratic equation $ax^2 - 4bx + 4c = 0$ (where $a > 0$) all lie in the segment $[2,3]$. Prove that:

   (a) $a \leq b \leq c < a + b$;

   (b) $\dfrac{a}{a+c} + \dfrac{b}{b+a} > \dfrac{c}{c+b}$.

4. Cross-shaped tiles  are to be placed on a 8×8 square grid without overlapping. Find the largest possible number of tiles that can be placed.

Second Day

5. On a mathematical olympiad, problem 5 for category D was worth 4 points. It turned out after the olympiad that the number of students who scored 3 points on this problem was equal to the number of those who scored 2 points on the problem. Each student scored at least 1 point on this problem. Given that the total number of points gained on this problem was 30 greater than the number of students, find the number of students who scored at least 3 points.

6. Consider the function $f(x) = \{x\} + \{1/x\}$.

   (a) Prove that $f(x) < 1.5$ for $x > 0$ and $f(x) < 2$ for $x < 0$.

   (b) Prove that for all $n \in \mathbb{N}$ there exists x_0 such that $f(x_0) > 2 - \dfrac{1}{n}$.

7. On the side AB of a triangle ABC with $BC < AC < AB$, points B_1 and C_2 are marked so that $AC_2 = AC$ and $BB_1 = BC$. Points B_2 on side AC and C_1 on the extension of CB are marked so that $CB_2 = CB$ and $CC_1 = CA$. Prove that the lines C_1C_2 and B_1B_2 are parallel.

8. Seven points are given on a plane, no three of which lie on a line. Any two of these points are joined by a segment. Is it possible to color these segments by several colors in such a way that, for each color, there are exactly three segments of that color and these three segments form a triangle?

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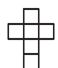
Category C

First Day

1. Find all pairs of integers (x,y) satisfying the equality
$$y(x^2+36)+x(y^2-36)+y^2(y-12)=0.$$

2. In a triangle ABC with a right angle at C, the altitude CD intersects the angle bisector AE at F. Lines ED and BF meet at G. Prove that the area of the quadrilateral $CEGF$ is equal to the area of the triangle BDF.

3. A set S consists of k sequences of $0,1,2$ of length n. For any two sequences $(a_i),(b_i)\in S$ we can construct a new sequence (c_i) such that $c_i=\left[\frac{a_i+b_i+1}{2}\right]$ and include it in S. Assume that after performing finitely many such operations we obtain all the 3^n sequences of $0,1,2$ of length n. Find the smallest possible value of k.

4. Cross-shaped tiles  are to be placed on a 9×9 square grid without overlapping. Find the largest possible number of tiles that can be placed.

Second Day

5. Find the number of pairs of positive integers (p,q) such that the roots of the equation $x^2-px-q=0$ do not exceed 10.

6. The equilateral triangles ABF and CAG are constructed in the exterior of a right-angled triangle ABC with $\angle C=90^\circ$. Let M be the midpoint of BC. Given that $MF=11$ and $MG=7$, find the length of BC.

7. Tom and Jerry play the following game. They alternately put pawns onto empty cells of a 20×20 square board. Tom plays first. A player wins if after his move some four pawns are at the vertices of a rectangle with sides parallel to the sides of the board. Determine who of them has a winning strategy.

8. Suppose that real numbers a,b,c,d satisfy
$$\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{d}+\frac{d}{c}+\frac{d}{a}+\frac{a}{d}=\frac{a}{c}+\frac{c}{a}+\frac{b}{d}+\frac{d}{b}+\frac{ac}{bd}+\frac{bd}{ac}+2.$$
Prove that at least two of the numbers are equal.

Category B

First Day

1. Find the locus of points M on the Cartesian plane such that the tangents from M to the parabola $y = x^2$ are perpendicular.

2. Find all pairs of positive integers (m,n) such that
$$(m-n)^2(n^2-m) = 4m^2n.$$

3. The diagonals of a convex quadrilateral $ABCD$ intersect at M. The bisector of $\angle ACD$ intersects the extension of BA over A at K. Prove that if $MA \cdot MC + MA \cdot CD = MB \cdot MD$, then $\angle BKC = \angle CDB$.

4. An equilateral triangle of side n is divided into n^2 equilateral triangles of side 1. Each of the vertices of the small triangles is labelled with number 1, except for one that is labelled with -1. Per move one can choose a line containing a side of a small triangle and change the signs of the numbers at points on this line. Determine all initial positions (the value of n and the position of the -1) for which one can achieve that all the labels equal 1 using the described operations.

Second Day

5. Tom and Jerry play the following game. They alternately put pawns onto empty cells of a 25×25 square board. Tom plays first. A player wins if after his move some four pawns are at the vertices of a rectangle with sides parallel to the sides of the board. Determine who of them has a winning strategy.

6. A rectangle $ABCD$ and a point X are given on plane.

   (a) Prove that among the segments XA, XB, XC, XD, some three are sides of a triangle.

   (b) Does (a) necessarily hold if $ABCD$ is a parallelogram?

7. Find all positive integers a and b such that $a^{a^a} = b^b$.

8. A set R of nonzero vectors on a plane is called *concordant* if it satisfies the following conditions:

   (i) For any vectors $a, b \in R$ (may be equal) the vector $S_b(a)$ symmetric to a with respect to the line perpendicular to b belongs to R;

   (ii) For any $a, b \in R$ there exists an integer k such that $a - S_b(a) = kb$.

   (a) Prove that for any two non-parallel and non-perpendicular vectors $a, b \in R$, either $a - b$ or $a + b$ is in R.

   (b) Does there exist an infinite concordant set? Find the largest possible cardinality of a finite concordant set.

Category A

First Day

1. Pit and Bill play the following game. First Pit writes down a number a, then Bill writes a number b, then Pit writes a number c. Can Pit always play so that the three equations

$$x^3+ax^2+bx+c=0, \quad x^3+bx^2+cx+a=0, \quad x^3+cx^2+ax+b=0$$

have: (a) a common real root; (b) a common negative root?

2. Find the number of pairs (n,q), where n is a positive integer and q a non-integer rational number with $0<q<2000$, that satisfy $\{q^2\}=\left\{\frac{n!}{2000}\right\}$.

3. Let $N\geq 5$ be given. Consider all sequences $(e_1,e_2,\dots,e_N)$ with each e_i equal to 1 or -1. Per move one can choose any five consecutive terms and change their signs. Two sequences are said to be *similar* if one of them can be transformed into the other in finitely many moves. Find the maximum number of pairwise non-similar sequences of length N.

4. The lateral sides and diagonals of a trapezoid intersect a line l, determining three equal segments on it. Must l be parallel to the bases of the trapezoid?

Second Day

5. Nine points are given on a plane, no three of which lie on a line. Any two of these points are joined by a segment. Is it possible to color these segments by several colors in such a way that, for each color, there are exactly three segments of that color and these three segments form a triangle?

6. A vertex of a tetrahedron is called *perfect* if the three edges at this vertex are sides of a certain triangle. How many perfect vertices can a tetrahedron have?

7. (a) Find all positive integers n for which the equation $(a^a)^n=b^b$ has a solution in positive integers a,b greater than 1.

   (b) Find all positive integers a,b satisfying $(a^a)^5=b^b$.

8. To any triangle with side lengths a,b,c and the corresponding angles α,β,γ (measured in radians), the 6-tuple $(a,b,c,\alpha,\beta,\gamma)$ is assigned. Find the minimum possible number n of distinct terms in 6-tuple assigned to a scalene triangle.