3-rd Bosnia and Hercegovina Mathematical Olympiad 1998

Sarajevo, May 16–17

First Day

1. Let P_1, P_2, P_3, P_4, P_5 be distinct points inside the figure D or on its boundary. Denote by M the minimum distance between dwo different points P_i. For which configuration of points P_i does M attain its maximum value, if

   (a) D is a unit square?

   (b) D is a unit equilateral triangle?

   (c) D is a unit circle?

2. If positive numbers x, y, z satisfy $x^2 + y^2 + z^2 = 1$, prove the inequality

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} \le \frac{3\sqrt{3}}{4}.$$

3. In a triangle ABC, the angle bisectors at A, B and C intersect the opposite sides at A_1, B_1, C_1, respectively. Point M lies on one of the segments A_1B_1, B_1C_1, C_1A_1, and M_1, M_2, M_3 are its orthogonal projections on the lines BC, CA, AB. Prove that one of the lengths MM_1, MM_2, MM_3 equals the sum of the other two.

Second Day

4. A circle of radius r is tangent to a line p at A. Let AB be the diameter of the circle and C be an arbitrary point on the circle other than A and B. Let D be the projection of C on AB and E be a point on the extension of CD over D with $ED = BC$. The tangents to the circle from point E intersect p at K and N. Prove that the length KN does not depend on the choice of C.

5. Show that if integers a, b, c and d satisfy $bc + ad = ac + 2bd = 1$, then they also satisfy $a^2 + c^2 = 2b^2 + 2d^2$.

6. The sequence $(u_n)_{n=0}^{\infty}$ is defined by $u_0 = 0$ and

$$u_{2n} = u_n, \quad u_{2n+1} = 1 - u_n \quad \text{for } n \in \mathbb{N}_0.$$

   (a) Determine u_{1998}.

   (b) If p is a natural number and $m = (2^p - 1)^2$, determine u_m.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
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