5-th Bosnia and Hercegovina Mathematical Olympiad 2000

Sarajevo, May 20–21, 2000

First Day

1. Determine real roots x_1, x_2 of the equation $x^5 - 55x + 21 = 0$, knowing that $x_1 x_2 = 1$.

2. In a triangle ABC, R denotes the circumradius and r the inradius. Let S be a point inside the triangle and let the lines AS, BS, CS meet the opposite sides at X, Y, Z, respectively. Show that

$$\frac{BX \cdot CX}{AX^2} + \frac{CY \cdot AY}{BY^2} + \frac{AZ \cdot BZ}{CZ^2} = \frac{R}{r} - 1$$

if and only if S is the incenter of the triangle.

3. A triple (x, y, z) of positive integers is called Pythagorean if $x \leq y \leq z$ and $x^2 + y^2 = z^2$. Prove that for every $n \in \mathbb{N}$ the number 2^{n+1} occurs in exactly n distinct Pythagorean triples.

Second Day

4. Prove that for any positive numbers a, b, c

$$\frac{bc}{a^2+2bc} + \frac{ca}{b^2+ca} + \frac{ab}{c^2+ab} \leq 1 \leq \frac{a^2}{a^2+2bc} + \frac{b^2}{b^2+ca} + \frac{c^2}{c^2+ab}.$$

5. Let T_n be the number of pairwise non-congruent triangles with perimeter m and integer side lengths. Prove that

   (a) $T_{1999} > T_{2000}$ and

   (b) $T_{4n+1} = T_{4n-2} + n$ for all $n \in \mathbb{N}$.

6. A triangle ABC with $\angle ABC = 3\angle CAB$ is given. Points M and N are taken on the side AC with N between A and M such that $\angle CBM = \angle MBN = \angle NBA$. Let L be an arbitrary interior point of segment BN and K be the point on BM such that $LK \parallel AC$. Show that the lines AL, NK and BC meet in a point.

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