Flanders Mathematical Olympiad 1995

Final Round

1. Four married couples meet for a game of chess. They draw lots to form four groups of two players. Knowing that

   > Beatrice plays against Eddy;
   >
   > Alice plays against Clara's husband;
   >
   > Freddy plays against Guy's wife;
   >
   > Doris plays against Alice's husband;
   >
   > Guy plays against Eddy's wife,

   can one tell who Hubert's wife is? Justify your answer.

2. How many real numbers $x \in [1,3]$ have the property that x^2 and x have the same fractional part?

3. Points a,b,c,d lie on the edge of a circular lake with radius R. The distances between a and c and between a and b are both equal to 500 meters. The times a swimmer needs to get from d to c, from d to a, and from a to b with the same constant velocity are in proportion $1 : 5 : 7$. Determine R.

4. (a) Prove that for all $n \in \mathbb{N}$ and all $\alpha \in [0, \pi]$:

   $$\sum_{k=0}^{n} \sin k\alpha = \frac{\sin \frac{(n+1)\alpha}{2} \sin \frac{n\alpha}{2}}{\sin \frac{\alpha}{2}} .$$

   (b) Let $G(n)$ be the average length of the diagonals of a regular n-gon ($n \geq 4$) inscribed in a circle of radius 1. Compute $\lim_{n \to \infty} G(n)$.