Flanders Mathematical Olympiad 1992

Final Round

1. For each positive integer n, determine the largest positive integer k such that 2^k divides $3^n + 1$.

2. Through its underground spy, the police was informed of a meeting place of a gang. The gang consists of five persons of different height, but the identities of the gang members are unknown and it is only known that the tallest of them is the leader. A policeman has the task to follow the leader.

   After the meeting, for security reasons, the gangsters leave the site one by one with 15-minute breaks between them. The policeman, thus being unable to spot the tallest one, decides to let the first two gangsters go and then to follow the first one who is taller than both of them. What's the probability that he will follow the correct person?

3. A sphere is inscribed in a cone with apotheme A. The tangent circle of the cone with the sphere determines the upper base of a cylinder that is inscribed in the sphere. Assume that the total area of the cone (including the base) equals nine times the area of a great circle of the sphere, and that the perimeter of the base of the cone is less than $2A$. Compute the height of the cylinder in terms of A.

4. Let A, B, P be positive numbers with $P \leq A + B$.

   (a) If θ_1, θ_2 are real numbers such that $A\cos\theta_1 + B\cos\theta_2 = P$, prove that

   $$A\sin\theta_1 + B\sin\theta_2 \leq \sqrt{(A+B-P)(A+B+P)}.$$

   (b) Show that equality occurs when $\theta_1 = \theta_2 = \theta$, where $\cos\theta = \frac{P}{A+B}$, $0 \leq \theta \leq \pi/2$.

   (c) Take $A = \dfrac{1}{2}ab$, $B = \dfrac{1}{2}cd$ and $P = \dfrac{1}{4}(a^2 + b^2 - c^2 - d^2)$ with $0 < a \leq b \leq a + c + d$, $c, d > 0$ and $c^2 + d^2 \leq a^2 + b^2$. Show that we the results in (a) and (b) can be interpreted as follows: from all quadrilaterals with successive side lengths a, b, c, d, the quadrangle with the largest area is the cyclic one.

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