Flanders Mathematical Olympiad 1991

Final Round

1. Show that the number $n = \overline{111\ldots11}$, consisting of 1991 digits 1, is not prime.

2. (a) Show that for every $n \in \mathbb{N}_0$ there is exactly one positive real number x such that $x^n + x^{n+1} = 1$. Denote this x by x_n.

   (b) Find $\lim\limits_{n\to+\infty} x_n$.

3. An equilateral triangle ABC and a point X on side AB different from A and B are given. Consider unique points Y and Z on sides BC and CA, respectively, such that $\triangle XYZ$ is equilateral. Assuming that the area of $\triangle XYZ$ is half the area of $\triangle ABC$, find the ratios $\dfrac{AX}{XB}$, $\dfrac{BY}{YC}$, $\dfrac{CZ}{ZA}$.

4. A word of length n consisting of digits 0 and 1 is called a *bit-string* of length n (for example, 000 and 01101 are bit-strings of length 3 and 5.) Define the sequence $s_1, s_2, \ldots$ of bit-strings of length $n \geq 2$ as follows:

   (1) s_1 is the bit-string 00...01, consisting of $n-1$ zeros and a 1;

   (2) s_{i+1} is obtained as follows:

   1° Remove the leftmost digit of s_i. This results in a bit-string t of length $n-1$.

   2° Examine whether the bit-string $t1$ (i.e. t followed by a 1) has already occured in $\{s_1, s_2, \ldots, s_i\}$. If it has not, take $s_{i+1} = t1$; otherwise, take $s_{i+1} = t0$.

   For example, for $n = 3$ we have $s_1 = 001 \to s_2 = 011 \to s_3 = 111 \to s_4 = 110 \to s_5 = 101 \to s_6 = 010 \to s_7 = 100 \to s_8 = 000 \to \ldots$)

   Prove that if $N = 2^n$, then the bit-strings $s_1, s_2, \ldots, s_N$ (all of length n) are all different.

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