Flanders Mathematical Olympiad 2006

Final Round

1. (a) Find all real numbers θ such that $\cos(4\theta) = \cos(3\theta)$.

   (b) Determine the integers a, b, c, d such that $\cos\frac{2\pi}{7}$, $\cos\frac{4\pi}{7}$, and $\cos\frac{6\pi}{7}$ are the solutions of the equation $ax^3 + bx^2 + cx + d = 0$.

2. Let P be a point on the side AB of an equilateral triangle ABC. Assume that the points $Q \in BC$, $R \in AC$, and $S \in AB$ are chosen in such a way that $PQ \perp AB$, $QR \perp BC$, and $RS \perp CA$. The points $Q' \in BC$, $R' \in AC$, and $S' \in AB$ are now chosen in such a way that $PQ' \perp BC$, $Q'R' \perp CA$, and $R'S' \perp AB$. If $S = S'$, determine $PB : AB$.

3. A total of 60 elfs and trolls are seated around a table. Trolls always lie, and all elfs always speak the truth, except when they make a little mistake. Everybody claims to sit between an elf and a troll, but exactly two elfs made a mistake! How many trolls are there at the table?

4. Find all functions $f : \mathbb{R} \setminus \{0,1\} \to \mathbb{R}$ such that

$$f(x) + f\left(\frac{1}{1-x}\right) = 1 + \frac{1}{x(1-x)}.$$

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