21-st Balkan Mathematical Olympiad

Pleven, Bulgaria – May 7, 2004

1. A sequence of real numbers $a_0, a_2, a_2, \ldots$ satisfies the condition

$$a_{m+n} + a_{m-n} - m + n - 1 = \frac{a_{2m} + a_{2n}}{2}$$

for all $m, n \in \mathbb{N}$ with $m \geq n$. If $a_1 = 3$, determine a_{2004}. *(Cyprus)*

2. Find all solutions in the set of prime numbers of the equation

$$x^y - y^x = xy^2 - 19.$$

(Albania)

3. Let O be an interior point of an acute-angled triangle ABC. The circles centered at the midpoints of the sides of the triangle ABC and passing through point O, meet in points K, L, M different from O. Prove that O is the incenter of the triangle KLM if and only if O is the circumcenter of the triangle ABC. *(Romania)*

4. A plane is divided into regions by a finite number of lines, no three of which are concurrent. We call two regions *neighboring* if their common boundary is either a segment, a ray, or a line. One should write an integer in each of the regions so as to fulfil the following two conditions:

   (i) The product of the numbers in two neighboring regions is less than their sum;

   (ii) The sum of all the numbers in the halfplane determined by any of the lines is equal to zero.

   Prove that this can be done if and only if not all the lines are parallel.

(Serbia and Montenegro)

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The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
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