6-th Balkan Mathematical Olympiad

Split, Yugoslavia – April 29 - May 6, 1989

1. Let $1 = d_1 < d_2 < \cdots < d_k = n$ be all divisors of a positive integer n. Find all n such that $k \geq 4$ and $d_1^2 + d_2^2 + d_3^2 + d_4^2 = n$. *(Bulgaria)*

2. Let $\overline{a_n \dots a_1 a_0} = 10^n a_n + \cdots + 10a_1 + a_0$ be the decimal representation of a prime number. If $n > 1$ and $a_n > 1$, prove that the polynomial
$$P(x) = a_n x^n + \cdots + a_1 x + a_0$$
is irreducible (over $\mathbb{Z}[x]$). *(Yugoslavia)*

3. A line l intersects the sides AB and AC of a triangle ABC at points B_1 and C_1, respectively, so that the vertex A and the centroid G of $\triangle ABC$ lie in the same half-plane determined by l. Prove that
$$S_{BB_1GC_1} + S_{CC_1GB_1} \geq \frac{4}{9} S_{ABC}.$$
(Greece)

4. Consider all families $\mathscr{F}$ of subsets of $\{1, 2, \dots, n\}$ which satisfy:
   - (i) If $A \in \mathscr{F}$, then $|A| = 3$;
   - (ii) If $A, B \in \mathscr{F}$ and $A \neq B$, then $|A \cap B| \leq 1$.

   Let $f(n)$ denote the maximum value of $|\mathscr{F}|$ over all such $\mathscr{F}$. Prove that
$$\frac{1}{6}(n^2 - 4n) \leq f(n) \leq \frac{1}{6}(n^2 - n).$$
(Romania)

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com