22-nd Austrian Mathematical Olympiad 1991

Final Round

Beginner Level – May 7

1. Suppose that a, b, and $\sqrt[3]{a}+\sqrt[3]{b}$ are rational numbers. Prove that $\sqrt[3]{a}$ and $\sqrt[3]{b}$ are also rational.

2. Solve in real numbers the equation
$$\frac{1}{x}+\frac{1}{x+2}-\frac{1}{x+4}-\frac{1}{x+6}-\frac{1}{x+8}-\frac{1}{x+10}+\frac{1}{x+12}+\frac{1}{x+14}=0.$$

3. Find the number of squares in the sequence given by $a_0=91$ and $a_{n+1}=10a_n+(-1)^n$ for $n\geq 0$.

4. Let AB be a chord of a circle k of radius r, with $AB=c$.
   (a) Construct triangle ABC with C on k in which a median from A or B is of a given length d.
   (b) For which c and d is this triangle unique?

Advanced Level

First Day – June 11

1. Consider a convex solid $\mathscr{K}$ in space and two parallel planes ε_1 and ε_2 on the distance 1 tangent to $\mathscr{K}$. A plane ε between ε_1 and ε_2 is on the distance d_1 from ε_1. Find all d_1 such that the part of $\mathscr{K}$ between ε_1 and ε always has a volume not exceeding half the volume of $\mathscr{K}$.

2. Find all functions $f:\mathbb{Z}\setminus\{0\}\to\mathbb{Q}$ satisfying
$$f\left(\frac{x+y}{3}\right)=\frac{f(x)+f(y)}{2}\quad \text{whenever } x,y,\frac{x+y}{3}\in\mathbb{Z}\setminus\{0\}.$$

3. (a) Prove that 91 divides $n^{37}-n$ for all integers n.
   (b) Find the largest k that divides $n^{37}-n$ for all integers n.

Second Day – June 12

4. The sequence (a_n) is given by $a_1=1$, $a_2=0$ and
$$a_{2k+1}=a_k+a_{k+1},\quad a_{2k+2}=2a_{k+1}\quad \text{for } k\in\mathbb{N}.$$
Find a_m for $m=2^{19}+91$.

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5. For all positive integers n prove the inequality

$$\left(\frac{1+(n+1)^{n+1}}{n+2}\right)^{n-1} > \left(\frac{1+n^n}{n+1}\right)^n .$$

6. Find the number of ten-digit natural numbers (which do not start with zero) containing no block 1991.

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