21-st Austrian Mathematical Olympiad 1990

Final Round

First Day – May 30

1. Determine the number of integers n with $1 \leq n \leq N = 1990^{1990}$ such that $n^2 - 1$ and N are coprime.

2. Show that for all integers $n \geq 2$, $\sqrt{2\sqrt[3]{3\sqrt[4]{4\dots\sqrt[n]{n}}}} < 2$.

3. In a convex quadrilateral $ABCD$, let E be the intersection point of the diagonals, and let F_1, F_2, and F be the areas of ABE, CDE, and $ABCD$, respectively. Prove that
$$\sqrt{F_1} + \sqrt{F_2} \leq \sqrt{F}.$$

Second Day – May 31

4. For each nonzero integer n find all functions $f : \mathbb{R} \setminus \{-3, 0\} \to \mathbb{R}$ satisfying
$$f(x+3) + f\left(-\frac{9}{x}\right) = \frac{(1-n)(x^2+3x-9)}{9n(x+3)} + \frac{2}{n} \quad \text{for all } x \neq 0, -3.$$
Furthermore, for each fixed n find all integers x for which $f(x)$ is an integer.

5. Determine all rational numbers r such that all solutions of the equation
$$rx^2 + (r+1)x + (r-1) = 0$$
are integers.

6. A convex pentagon $ABCDE$ is inscribed in a circle. The distances of A from the lines BC, CD, DE are a, b, c, respectively. Compute the distance of A from the line BE.

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