20-th Austrian Mathematical Olympiad 1989

Final Round

Beginner Level

1. Natural numbers $a \leq b \leq c \leq d$ satisfy $a+b+c+d=30$. Find the maximum value of the product $P = abcd$.

2. If a and b are nonnegative real numbers with $a^2+b^2=4$, show that
$$\frac{ab}{a+b+2} \leq \sqrt{2}-1$$
and determine when equality occurs.

3. Let a be a real number. Prove that if the equation $x^2-ax+a=0$ has two real roots x_1 and x_2, then $x_1^2+x_2^2 \geq 2(x_1+x_2)$.

4. Prove that for any triangle each exradius is less than four times the circumradius.

Advanced Level

First Day

1. Consider the set S_n of all the 2^n numbers of the type $2 \pm \sqrt{2 \pm \sqrt{2 \pm \dots}}$, where number 2 appears $n+1$ times.
   (a) Show that all members of S_n are real.
   (b) Find the product P_n of the elements of S_n.

2. Find all triples (a,b,c) of integers with $abc = 1989$ and $a+b-c = 89$.

3. Show that it is possible to situate eight parallel planes at equal distances such that each plane contains precisely one vertex of a given cube. How many such configurations of planes are there?

Second Day

4. We are given a circle k and nonparallel tangents t_1, t_2 at points P_1, P_2 on k, respectively. Lines t_1 and t_2 meet at A_0. For a point P_3 on the smaller arc P_1P_2, the tangent t_3 to k at P_3 meets t_1 at A_1 and t_2 and A_2. How must P_3 be chosen so that the triangle $A_0A_1A_2$ has maximum area?

5. Find all real solutions of the system
$$\begin{aligned} x^2+2yz &= x, \\ y^2+2zx &= y, \\ z^2+2xy &= z. \end{aligned}$$

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6. Determine all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(f(n)) + f(n) = 2n + 6$ for all $n \in \mathbb{N}_0$.

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