18-th Austrian–Polish Mathematical Competition 1995

Hollabrunn, Austria

Individual Competition – June 28–29

First Day

1. Determine all real solutions $(a_1,\dots,a_n)$ of the following system of equations:

$$\begin{cases} a_3 = & a_2 + a_1 \\ a_4 = & a_3 + a_2 \\ & \dots \\ a_n = & a_{n-1} + a_{n-2} \\ a_1 = & a_n + a_{n-1} \\ a_2 = & a_1 + a_n. \end{cases}$$

2. Let $X = \{A_1, A_2, A_3, A_4\}$ be a set of four distinct points in the plane. Show that there exists a subset Y of X with the property that there is no (closed) disk K such that $K \cap X = Y$.

3. Let $P(x) = x^4 + x^3 + x^2 + x + 1$. Show that there exist two non-constant polynomials $Q(y)$ and $R(y)$ with integer coefficients such that for all

$$Q(y) \cdot R(y) = P(5y^2) \quad \text{for all } y.$$

Second Day

4. Determine all polynomials $P(x)$ with real coefficients such that

$$P(x)^2 + P\left(\frac{1}{x}\right)^2 = P(x^2)P\left(\frac{1}{x^2}\right) \quad \text{for all } x.$$

5. In an equilateral triangle ABC, A_1, B_1, C_1 are the midpoints of the sides BC, CA, AB, respectively. Three parallel lines p, q and r pass through A_1, B_1 and C_1 and intersect the lines B_1C_1, C_1A_1 and A_1B_1 at points A_2, B_2, C_2, respectively. Prove that the lines AA_2, BB_2, CC_2 have a common point D which lies on the circumcircle of the triangle ABC.

6. The Alpine Club organizes four mountain trips for its n members. Let E_1, E_2, E_3, E_4 be the teams participating in these trips. In how many ways can these teams be formed so as to satisfy

$$E_1 \cap E_2 \neq \emptyset, \quad E_2 \cap E_3 \neq \emptyset, \quad E_3 \cap E_4 \neq \emptyset?$$

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

Team competition – June 30

7. Consider the equation $3y^4+4cy^3+2xy+48=0$, where c is an integer parameter. Determine all values of c for which the number of integral solutions (x,y) satisfying the conditions (i) and (ii) is maximal:

   (i) $|x|$ is a square of an integer;

   (ii) y is a squarefree number.

8. Consider the cube with the vertices at the points $(\pm1,\pm1,\pm1)$. Let $V_1,\dots,V_{95}$ be arbitrary points within this cube. Denote $v_i=\overrightarrow{OV_i}$, where $O=(0,0,0)$ is the origin. Consider the 2^{95} vectors of the form $s_1v_1+s_2v_2+\cdots+s_{95}v_{95}$, where $s_i=\pm1$.

   (a) If $d=48$, prove that among these vectors there is a vector $w=(a,b,c)$ such that $a^2+b^2+c^2\le 48$.

   (b) Find a smaller d (the smaller, the better) with the same property.

9. Prove that for all positive integers n,m and all real numbers $x,y>0$ the following inequality holds:

$$(n-1)(m-1)(x^{n+m}+y^{n+m})+(n+m-1)(x^ny^m+x^my^n)\ge nm(x^{n+m-1}y+xy^{n+m-1}).$$

www.imomath.com

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com