17-th Austrian–Polish Mathematical Competition 1994

Pogorzel Warszawska, Poland

Individual Competition – June 29–30

First Day

1. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the conditions
$$f(x+19) \le f(x)+19 \quad \text{and} \quad f(x+94) \ge f(x)+94 \quad \text{for all } x \in \mathbb{R}.$$
Prove that $f(x+1) = f(x)+1$ for all $x \in \mathbb{R}$.

2. The sequences (a_n) and c_n are given by $a_0 = \frac{1}{2}$, $c_0 = 4$, and for $n \ge 0$
$$a_{n+1} = \frac{2a_n}{1+a_n^2}, \quad c_{n+1} = c_n^2 - 2c_n + 2.$$
Prove that for all $n \ge 1$, $a_n = \dfrac{2c_0c_1\cdots c_{n-1}}{c_n}$.

3. A rectangular building consists of 30 square rooms situated like the cells of a 2×15 board. In each room there are three doors, each of which leads to another room (not necessarily different). How many ways are there to distribute the doors between the rooms so that it is possible to get from any room to any other one without leaving the building?

Second Day

4. The vertices of a regular $n+1$-gon are denoted by $P_0, P_1, \dots, P_n$ in some order ($n \ge 2$). Each side of the polygon is assigned a natural number as follows: if the endpoints of the side are P_i and P_j, then the assigned number equals $|i-j|$. Let S be the sum of all $n+1$ assigned numbers.
   (a) Given n, what is the smallest possible value of S?
   (b) If P_0 is fixed, how many different assignments are there for which S attains the smallest value?

5. Find all integral solutions of the equation
$$\frac{1}{2}(x+y)(y+z)(z+x) + (x+y+z)^3 = 1 - xyz.$$

6. Let $n > 1$ be an odd positive integer. Assume that positive integers $x_1, x_2, \dots, x_n \ge 0$ satisfy
$$\begin{aligned}
(x_2-x_1)^2 + 2(x_2+x_1) + 1 &= n^2 \\
(x_3-x_2)^2 + 2(x_3+x_2) + 1 &= n^2 \\
\cdots\cdots\cdots\cdots\cdots \\
(x_1-x_n)^2 + 2(x_1+x_n) + 1 &= n^2.
\end{aligned}$$

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Show that there exists j, $1 \leq j \leq n$, such that $x_j = x_{j+1}$. Here $x_{n+1} = x_1$.

Team competition – July 1

7. Determine all two-digit positive integers $n = \overline{ab}$ (in the decimal system) with the property that for all integers x the difference $x^a - x^b$ is divisible by n.

8. Given real numbers a, b, find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x,y) = af(x,z) + bf(y,z) \quad \text{for all } x,y,z \in \mathbb{R}.$$

9. On the plane are given four distinct points A, B, C, D on a line g in this order, at the mutual distances $AB = a$, $BC = b$, $CD = c$.

   (a) Construct (if possible) a point P outside line g such that $\angle APB = \angle BPC = \angle CPD$.

   (b) Prove that such a point P exists if and only if $(a+b)(b+c) < 4ac$.

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