16-th Austrian–Polish Mathematical Competition 1993

Graz, Austria

Individual Competition – June 30–July 1

First Day

1. Solve in positive integers x, y the equation $2^x - 3^y = 7$.

2. Consider all tetrahedra $ABCD$ in which the sum of the areas of the faces ABD, ACD, BCD does not exceed 1. Among such tetrahedra, find those with the maximum volume.

3. Define $f(n) = n+1$ if $n = p^k > 1$ is a power of a prime number, and $f(n) = p_1^{k_1} + \cdots + p_r^{k_r}$ for natural numbers $n = p_1^{k_1} \cdots p_r^{k_r}$ ($r > 1$, $k_i > 0$). Given $m > 1$, we construct the sequence $a_0 = m$, $a_{j+1} = f(a_j)$ for $j \geq 0$ and denote by $g(m)$ the smallest term in this sequence. For each $m > 1$, determine $g(m)$.

Second Day

4. The Fibonacci sequence is defined by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Positive integers A and B are such that B^{93} is divisible by A^{19} and A^{93} is divisible by B^{19}. Prove that for all integers $n \geq 1$, $(A^4 + B^8)^{F_{n+1}}$ is divisible by $(AB)^{F_n}$.

5. Solve in real numbers the system
$$\begin{cases} x^3 + y &= 3x + 4 \\ 2y^3 + z &= 6y + 6 \\ 3z^3 + x &= 9z + 8. \end{cases}$$

6. If $a, b \geq 0$ are real numbers, prove the inequality
$$\begin{aligned} \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 &\leq \frac{a + \sqrt[3]{a^2b} + \sqrt[3]{ab^2} + b}{4} \\ &\leq \frac{a + \sqrt{ab} + b}{3} \leq \sqrt{\left(\frac{\sqrt[3]{a^2} + \sqrt[3]{b^2}}{2}\right)^3}. \end{aligned}$$
For each of the inequalities, find the cases of equality.

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Team competition – July 2

7. The sequence (a_n) is defined by $a_0 = 0$ and $a_{n+1} = [\sqrt[3]{a_n + n}]^3$ for $n \geq 0$.

   (a) Find a_n in terms of n.

   (b) Find all n for which $a_n = n$.

8. Determine all real polynomials $P(z)$ for which there exists a unique real polynomial $Q(x)$ satisfying the conditions

$$Q(0) = 0, \quad x + Q(y + P(x)) = y + Q(x + P(y)) \quad \text{for all } x, y \in \mathbb{R}.$$

9. Point P is taken on the extension of side AB of an equilateral triangle ABC so that A is between B and P. Denote by a the side length of triangle ABC, by r_1 the inradius of triangle PAC, and by r_2 the exradius of triangle PBC opposite P. Find the sum $r_1 + r_2$ as a function in a.

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