15-th Austrian–Polish Mathematical Competition 1992

Nowy Sącz, Poland

Individual Competition – June 22–23

First Day

1. For a natural number n, denote by $s(n)$ the sum of all positive divisors of n. Prove that for every $n > 1$ the product $s(n-1)s(n)s(n+1)$ is even.

2. Each point on the boundary of a square has to be colored in one color. Consider all right triangles with the vertices on the boundary of the square. Determine the least number of colors for which there is a coloring such that no such triangle has all its vertices of the same color.

3. For all positive numbers a,b,c prove the inequality
$$2\sqrt{bc+ca+ab} \leq \sqrt{3}\cdot\sqrt[3]{(b+c)(c+a)(a+b)}.$$

Second Day

4. Let k be a positive integer and u,v be real numbers. Consider
$$P(x) = (x-u^k)(x-uv)(x-v^k) = x^3+ax^2+bx+c.$$
   (a) For $k=2$ prove that if a,b,c are rational then so is uv.
   (b) Is that also true for $k=3$?

5. Given a circle k with center M and radius r, let AB be a fixed diameter of k and let K be a fixed point on the segment AM. Denote by t the tangent ot k at A. For any chord CD through K other than AB, denote by P and Q the intersection points of BC and BD with t, respectively. Prove that $AP\cdot AQ$ does not depend on CD.

6. A function $f:\mathbb{Z}\to\mathbb{Z}$ has the following properties:
$$\begin{aligned} f(92+x) &= f(92-x) \\ f(19\cdot 92+x) &= f(19\cdot 92-x) \quad (19\cdot 92 = 1748) \\ f(1992+x) &= f(1992-x) \end{aligned}$$
for all integers x. Can all positive divisors of 92 occur as values of f?

Team competition – June 24

7. Consider triangles ABC in space.

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(a) What condition must the angles α,β,γ of $\triangle ABC$ fulfil in order that there is a point P in space such that $\angle APB$, $\angle BPC$, $\angle CPA$ are right angles?

(b) Let d be the longest of the edges PA, PB, PC and let h be the longest altitude of $\triangle ABC$. Show that $\frac{1}{3}\sqrt{6}h \le d \le h$.

8. Let $n \ge 3$ be a given integer. Nonzero real numbers $a_1, \dots, a_n$ satisfy:

$$\frac{-a_1-a_2+a_3+\cdots+a_n}{a_1} = \frac{a_1-a_2-a_3+a_4+\cdots+a_n}{a_2} = \cdots$$
$$\cdots = \frac{a_1+\cdots+a_{n-2}-a_{n-1}-a_n}{a_{n-1}} = \frac{-a_1+a_2+\cdots+a_{n-1}-a_n}{a_n}.$$

What values can be taken by the product

$$\frac{a_2+\cdots+a_n}{a_1} \cdot \frac{a_1+a_3+\cdots+a_n}{a_2} \cdot \dots \cdot \frac{a_1+\cdots+a_{n-1}}{a_n}?$$

9. Given an integer $n > 1$, consider words composed of n letters A and n letters B. A word $X_1 \dots X_{2n}$ is said to belong to set $R(n)$ (respectively, $S(n)$) if no initial segment (respectively, exactly one initial segment) $X_1 \dots X_k$ with $1 \le k < 2n$ consists of equally many letters A and B. If $r(n)$ and $s(n)$ denote the cardinalities of $R(n)$ and $S(n)$ respectively, compute $s(n)/r(n)$.

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