10-th Austrian–Polish Mathematical Competition 1987

Austria

Individual Competition

First Day

1. Three pairwise orthogonal chords of a sphere $\mathscr{S}$ are drawn through a given point P inside $\mathscr{S}$. Prove that the sum of the squares of their lengths does not depend on their directions.

2. Let n be the square of an integer whose each prime divisor has an even number of decimal digits. Consider $P(x) = x^n - 1987x$. Show that if x, y are rational numbers with $P(x) = P(y)$, then $x = y$.

3. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+1) = f(x)+1$ for all x. Given $a \in \mathbb{R}$, define the sequence (x_n) recursively by $x_0 = a$ and $x_{n+1} = f(x_n)$ for $n \geq 0$. Suppose that, for some positive integer m, the difference $x_m - x_0 = k$ is an integer. Prove that the limit $\lim\limits_{n\to\infty} \dfrac{x_n}{n}$ exists and determine its value.

Second Day

4. Is there a 2000-element subset A of $\{1, 2, \dots, 3000\}$ with the property that $2x \notin A$ whenever $x \in A$?

5. The Euclidian three-dimensional space has been partitioned into three nonempty sets A_1, A_2, A_3. Show that one of these sets contains, for each $d > 0$, a pair of points at mutual distance d.

6. Let C be a unit circle and $n \geq 1$ be a fixed integer. For any set A of n points $P_1, \dots, P_n$ on C define
$$D(A) = \max_d \min_i \delta(P_i, d),$$
where d goes over all diameters of C and $\delta(P, l)$ denotes the distance from point P to line l. Let $\mathscr{F}_n$ be the family of all such sets A. Determine $D_n = \min\limits_{A \in \mathscr{F}_n} D(A)$ and describe all sets A with $D(A) = D_n$.

Team competition

7. For any natural number $n = \overline{a_k \dots a_1 a_0}$ ($a_k \neq 0$) in decimal system write
$$p(n) = a_0 \cdot a_1 \cdot \dots \cdot a_k, \quad s(n) = a_0 + a_1 + \dots + a_k, \quad n^* = \overline{a_0 a_1 \dots a_k}.$$
Consider $P = \{n \mid n = n^*, \ \frac{1}{3} p(n) = s(n) - 1\}$ and let Q be the set of numbers in P with all digits greater than 1.

(a) Show that P is infinite.

(b) Show that Q is finite.

(c) Write down all the elements of Q.

8. A circle of perimeter 1 has been dissected into four equal arcs B_1, B_2, B_3, B_4. A closed smooth non-selfintersecting curve C has been composed of translates of these arcs (each B_i possibly occurring several times). Prove that the length of C is an integer.

9. Let M be the set of all points (x, y) in the cartesian plane, with integer coordinates satisfying $1 \leq x \leq 12$ and $1 \leq y \leq 13$.

   (a) Prove that every 49-element subset of M contains four vertices of a rectangle with sides parallel to the coordinate axes.

   (b) Give an example of a 48-element subset of M without this property.

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