9-th Austrian–Polish Mathematical Competition 1986

Wrocław, Poland

Individual Competition – June 25–26

First Day

1. A non-rectangular triangle $A_1A_2A_3$ is given. Circles Γ_1 and Γ_2 are tangent at A_3, Γ_2 and Γ_3 are tangent at A_1, and Γ_3 and Γ_1 are tangent at A_2. Points O_1, O_2, O_3 are the centers of $\Gamma_1, \Gamma_2, \Gamma_3$, respectively. Supposing that the triangles $A_1A_2A_3$ and $O_1O_2O_3$ are similar, determine their angles.

2. The monic polynomial $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ of degree $n > 1$ has n distinct negative roots. Prove that
$$a_1P(1) > 2n^2a_0.$$

3. Each point in space is colored either blue or red. Show that there exists a unit square having exactly 0, 1 or 4 blue vertices.

Second Day

4. Find all triples of positive integers (x, y, z) such that
$$x^{z+1} - y^{z+1} = 2^{100}.$$

5. Find all real solutions of the system of equations
$$\begin{cases} x^2 + y^2 + u^2 + v^2 &= 4 \\ xu + yv + xv + yu &= 0 \\ xyu + yuv + uvx + vxy &= -2 \\ xyuv &= -1. \end{cases}$$

6. Let $\mathscr{M}$ be the set of all tetrahedra whose inscribed and circumscribed spheres are concentric. If the radii of these spheres are denoted by r and R respectively, find the possible values of R/r over all tetrahedra from $\mathscr{M}$.

Team competition – June 27

7. Let k and n be integers with $0 < k \leq n^2/4$ such that k has no prime divisor greater than n. Prove that k divides $n!$.

8. Pairwise distinct real numbers are arranged into an $m \times n$ rectangular array. In each row the entries are arranged increasingly from left to right. Each column is then rearranged in decreasing order from top to bottom. Prove that in the reorganized array, the rows remain arranged increasingly.

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9. Find all continuous monotonic functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(1) = 1$ and $f(f(x)) = f(x)^2$ for all $x \in \mathbb{R}$.

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