8-th Austrian–Polish Mathematical Competition 1985

Hollabrunn, Austria

Individual Competition – June 25–26

First Day

1. Prove that if a,b,c are distinct nonzero numbers with $a+b+c=0$, then
$$\left(\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{C}\right)\left(\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}\right)=9.$$

2. There are n persons $P_1,P_2,\dots,P_n$ at a party. Assume that $P_1,P_2,\dots,P_{n-6}$ know $4,5,\dots,n-3$ persons, respectively, that P_{n-5},P_{n-4},P_{n-3} know $n-2$ persons each, and that P_{n-2},P_{n-1},P_n know $n-1$ persons each. ("Knowing" is a symmetric relation, and noone is assumed to know himself/herself.) Find all $n\geq 8$ for which this is possible.

3. Prove that in a convex quadrilateral of area 1 the sum of the lengths of all sides and diagonals is not less than $4+\sqrt{8}$.

Second Day

4. Find all real solutions (x,y) of the system of equations
$$\begin{cases} x^4+y^2-xy^3-\frac{9}{8}x=0,\\ y^4+x^2-yx^3-\frac{9}{8}y=0. \end{cases}$$

5. We are given a set of weights consisting of several identical groups of four weights of different (positive) integer masses. Using these weights we are able to weigh every integer mass not exceeding 1985. In how many ways can one compose such a set with the smallest possible total mass?

6. For a point P inside a tetrahedron $ABCD$, points S_A,S_B,S_C,S_D denote the centroids of the tetrahedra $PBCD,PCDA,PDAB,PABC$, respectively. Show that the volume of the tetrahadron $S_AS_BS_CS_D$ equals $\frac{1}{64}$ the volume of $ABCD$.

Team competition – June 27

7. If x_1,x_2,x_3,x_4 are real numbers, not all zero, find an upper bound for
$$\frac{x_1x_2+2x_2x_3+x_3x_4}{x_1^2+x_2^2+x_3^2+x_4^2}.$$
The smaller bound, the better the solution.

8. A convex n-gon $A_0A_1\dots A_{n-1}$ has been partitioned into $n-2$ triangles by nonintersecting diagonals. Show that these triangles can be labelled $\triangle_1, \triangle_2, \dots, \triangle_{n-2}$ in such a way that A_i is a vertex of $\triangle_i$ for all i, and find the number of such labellings.

9. Given a convex polygon, prove that there exists a point Q inside the polygon and three vertices A_1, A_2, A_3 such that each ray A_iQ ($i = 1,2,3$) makes acute angles with the two sides emanating from A_i.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com