# 7-th Austrian–Polish Mathematical Competition 1984

## Poznań, Poland

**Individual Competition** – July 4–5

*First Day*

1. Prove that if the feet of the altitudes of a tetrahedron are the incenters of the corresponding faces, then the tetrahedron is regular.

2. Let $A$ be the set of four-digit natural numbers having exactly two distinct digits, none of which is zero. Interchanging the two digits of $n \in A$ yields a number $f(n) \in A$ (for instance, $f(3111) = 1333$). Find those $n \in A$ with $n > f(n)$ for which $\gcd(n, f(n))$ is the largest possible.

3. For all positive numbers $a, x_1, \dots, x_n$ ($n \geq 2$) prove the inequality
$$\frac{a^{x_1-x_2}}{x_1+x_2} + \frac{a^{x_2-x_3}}{x_2+x_3} + \cdots + \frac{a^{x_n-x_1}}{x_n+x_1} \geq \frac{n^2}{2(x_1+\cdots+x_n)}$$
and find the cases of equality.

*Second Day*

4. A regular heptagon $A_1A_2\cdots A_7$ is inscribed in circle $\mathscr{C}$. Point $P$ is taken on the shorter arc $A_7A_1$. Prove that
$$PA_1 + PA_3 + PA_5 + PA_7 = PA_2 + PA_4 + PA_6.$$

5. Given $n > 2$ nonnegative distinct integers $a_1, \dots, a_n$, find all nonnegative integers $y$ and $x_1, \dots, x_n$ satisfying $\gcd(x_1, \dots, x_n) = 1$ and
$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &= yx_1 \\ a_2x_1 + a_3x_2 + \cdots + a_1x_n &= yx_2 \\ \cdots\cdots\cdots & \quad \cdots \\ a_nx_1 + a_1x_2 + \cdots + a_{n-1}x_n &= yx_n. \end{aligned}$$

6. In a dancing hall, there are $n$ girls standing in one row and $n$ boys in the other row across them (so that all $2n$ dancers form a $2 \times n$ board). Each dancer gives her/his left hand to a neighboring person standing to the left, across, or diagonally to the left. The analogous rule applies for right hands. No dancer gives both hands to the same person. In how many ways can the dancers do this?

**Team competition** – July 6



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7. A matrix

$$(a_{ij}) = \begin{matrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{matrix}$$

of real numbers satisfies $|a_{ij}| \le 1$ and $\sum_{i=1}^{m} a_{ij} = 0$ for all $j$. Show that one can permute the entries in each column in such a way that the obtained matrix $(b_{ij})$ satisfies $\left|\sum_{j=1}^{n} b_{ij}\right| < 2$ for all $i$.

8. The functions $f, g : (1,\infty) \to (1,\infty)$ are given by $f_0(x) = 2x$ and $f_1(x) = \frac{x}{x-1}$. Show that for any real numbers $a, b$ with $1 \le a < b$ there exist a positive integer $k$ and indices $i_1, i_2, \dots, i_k \in \{0,1\}$ such that

$$a < f_{i_k}(f_{i_{k-1}}(\dots(f_{i_1}(2))\dots)) < b.$$

9. Find all functions $f : \mathbb{Q} \to \mathbb{R}$ satisfying

$$f(x+y) = f(x)f(y) - f(xy) + 1 \quad \text{for all } x, y \in \mathbb{Q}.$$