4-th Austrian–Polish Mathematical Competition 1981

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Individual Competition – June ??

First Day

1. Find the least integer $k > 16$ for which the set $\{16, 17, \dots, k\}$ contains 15 distinct integers $b_1, b_2, \dots, b_{15}$ such that b_m is divisible by m for $1 \le m \le 15$.

2. The sequence $a_0, a_1, a_2, \dots$ is defined by $a_{n+1} = a_n^2 + (a_n - 1)^2$ for $n \ge 0$. Find all rational numbers a_0 for which there exist four distinct indices k, m, p, q such that $a_q - a_p = a_m - a_k$.

3. In a triangle ABC, r is the inradius, r_A the radius of the circle touching segments AB, AC and the incircle of $\triangle ABC$, and r_B and r_C are defined analogously. Prove that
$$r_A + r_B + r_C \ge r,$$
equality holding if and only if $\triangle ABC$ is equilateral.

Second Day

4. Let $n \ge 3$ cells be arranged into a circle. Each cell can be occupied by 0 or 1. The following operation is admissible: Choose a any cell C occupied by a 1, change it into a 0 and simultaneously reverse the entries in the two cells adjacent to C (so that x, y become $1-x, 1-y$). Initially, there is a 1 in one cell and zeros elsewhere. For which values of n is it possible to obtain zeros in all cells in a finite number of admissible steps?

5. Let $P(x) = x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$ be a polynomial with rational coefficients. Show that if $P(x)$ has exactly one real root ξ, then ξ is a rational number.

6. The sequences (x_n), (y_n), (z_n) are given by
$$x_{n+1} = y_n + \frac{1}{x_n}, \quad y_{n+1} = z_n + \frac{1}{y_n}, \quad z_{n+1} = x_n + \frac{1}{z_n} \quad \text{for } n \ge 0,$$
where x_0, y_0, z_0 are given positive numbers. Prove that these sequences are unbounded.

Team competition – June ??

7. Let $a > 3$ be an odd integer. Show that for every positive integer n the number $a^{2^n} - 1$ has at least $n+1$ distinct prime divisors.

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8. The plane has been partitioned into N regions by three bunches of parallel lines. What is the least number of lines needed in order that $N > 1981$?

9. For a function $f : [0,1] \to [0,1]$ we define $f^1 = f$ and $f^{n+1}(x) = f(f^n(x))$ for $0 \le x \le 1$ and $n \in \mathbb{N}$. Given that there is an n such that

$$|f^n(x) - f^n(y)| < |x - y| \quad \text{for all distinct } x, y \in [0,1],$$

prove that there is a unique $x_0 \in [0,1]$ such that $f(x_0) = x_0$.

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