9-th Asian–Pacific Mathematical Olympiad 1997

1. Given

$$S = 1 + \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}+\frac{1}{6}} + \cdots + \frac{1}{1+\frac{1}{3}+\frac{1}{6}+\cdots+\frac{1}{1993006}},$$

   where the denominators contain partial sums of the sequence of reciprocals of triangular numbers (i.e. $k = \frac{n(n+1)}{2}$ for $n = 1, 2, \dots, 1996$). Prove that $S > 1001$.

2. Find an integer n with $100 \le n \le 1997$ such that $\dfrac{2^n + 2}{n}$ is also an integer.

3. In a triangle ABC, let m_a, m_b, m_c be the lengths of the angle bisectors (internal to the triangle), M_a, M_b, M_c the lengths of the angle bisectors extended until they meet the circle, and $l_a = \dfrac{m_a}{M_a}$, $l_b = \dfrac{m_b}{M_b}$, $l_c = \dfrac{m_c}{M_c}$. Prove that

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \ge 3,$$

   and that equality holds if and only if ABC is an equilateral triangle.

4. Triangle $A_1A_2A_3$ has a right angle at A_3. A sequence of points is defined by the following iterative process, where n is a positive integer. From A_n ($n \ge 3$), a perpendicular line is drawn to meet $A_{n-2}A_{n-1}$ at A_{n+1}.

   (a) Prove that one and only one point P is interior to every triangle $A_{n-2}A_{n-1}A_n$, $n \ge 3$.

   (b) Let A_1 and A_3 be fixed, and let A_2 assume all possible locations. Find the locus of P.

5. Suppose that n people $A_1, A_2, \dots, A_n$, ($n \ge 3$) are seated in a circle and that A_i has a_i objects such that $a_1 + a_2 + \cdots + a_n = nN$, where N is a positive integer. In order that each person has the same number of objects, each person A_i is to give or to receive a certain number of objects to or from its two neighbours A_{i-1} and A_{i+1}. (Here A_{n+1} means A_1 and A_0 means A_n.) How should this redistribution be performed so that the total number of objects transferred is minimum?

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