6-th Asian–Pacific Mathematical Olympiad 1994

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that:

   (i) $f(x) + f(y) + 1 \geq f(x+y) \geq f(x) + f(y)$ for all $x, y \in \mathbb{R}$,

   (ii) $f(0) \geq f(x)$ for all x with $0 \leq x < 1$,

   (iii) $-f(-1) = f(1) = 1$.

2. Given a nondegenerate triangle ABC, with circumcenter O, orthocenter H, and circumradius R, prove that $OH < 3R$.

3. Let n be an integer of the form $a^2 + b^2$, where a and b are coprime integers, such that if $p \leq \sqrt{n}$ is a prime, then p divides ab. Determine all such n.

4. Is there an infinite set of points in the plane such that no three points are collinear, and the distance between any two points is rational?

5. You are given three lists A, B, and C. List A contains the numbers of the form 10^k in base 10, with k any positive integer. Lists B and C contain the same numbers translated into base 2 and 5 respectively:

A	B	C
10	1010	20
100	1100100	400
1000	1111101000	13000
$\vdots$	$\vdots$	$\vdots$

   Prove that for every integer $n > 1$, there is exactly one number in exactly one of the lists B or C that has exactly n digits.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com