14-th Asian–Pacific Mathematical Olympiad 2002

1. Let $a_1, a_2, \dots, a_n$ be a sequence of non-negative integers, where n is a positive integer. Let $A_n = \dfrac{a_1 + a_2 + \cdots + a_n}{n}$. Prove that
$$a_1! a_2! \cdots a_n! \geq \left([A_n]!\right)^n .$$
When does equality hold?

2. Find all positive integers a and b such that $\dfrac{a^2+b}{b^2-a}$ and $\dfrac{b^2+a}{a^2-b}$ are both integers.

3. Let ABC be an equilateral triangle. Let P be a point on the side AC and Q be a point on the side AB so that both triangles ABP and ACQ are acute. Let R and S be the orthocenters of triangles ABP and ACQ respectively. Segments BP and CQ meet at point T. If triangle TRS is equilateral, find all possible values of $\angle CBP$ and $\angle BCQ$.

4. Let x, y, z be positive numbers such that $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$. Show that
$$\sqrt{x+yz} + \sqrt{y+zx} + \sqrt{z+xy} \geq \sqrt{xyz} + \sqrt{x} + \sqrt{y} + \sqrt{z}.$$

5. Find all functions f from $\mathbb{R}$ to $\mathbb{R}$ satisfying:
   (i) There are only finitely many $s \in \mathbb{R}$ for which $f(s) = 0$, and
   (ii) $f(x^4 + y) = x^3 f(x) + f(f(y))$ for all $x, y \in \mathbb{R}$.

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